A link between information theory and SPH

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3[rd] Iberian Congress Advances on SPH

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Quick SPH

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- some definition of V_i
- some "closure" to get the pressures p_i

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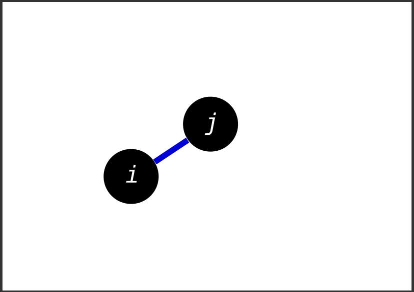
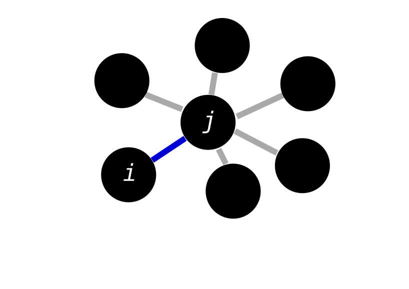
Forget SPH – SFs

All we have is *particles*.
Particle i influences particle j through a shape function (SF), "$\phi_i(\boldsymbol{r}_j)$".



Forget SPH – SFs

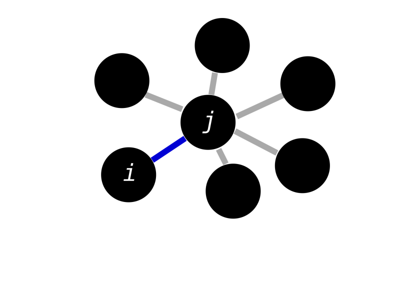
All we have is *particles*.
Particle i influences particle j through a shape function (SF), "$\phi_i(\boldsymbol{r}_j)$".
Subject to constraints, such as $\sum_i \phi_i(\boldsymbol{r}_j) = 1$.



Information entropy

Too many possibilities! But, information theory's "unbiased inference" (Jaynes, 1957) says we should maximize

$$S_j = -\sum_i \phi_i(\boldsymbol{r}_j) \log \left[\phi_i(\boldsymbol{r}_j)\right].$$



SPH?

The resulting SFs are very wide. Hence, Arroyo and Ortiz (2006) added an energy to make SFs more local:

$$E_j = \beta \sum_i \phi_i(\boldsymbol{r}_j) r_{ij}^2.$$

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If we forget constraints, the extreme of $F = E - TS$ gives

$$\phi_i(\boldsymbol{r}_j) = \exp(-\beta r_{ij}^2).$$

More than enough to drive me(*) nuts at the time (*from* that time, to be precise.)

Building a theory: ideal part

A continuum theory already exists[1], so we may inspire ourselves.

$$-S = c^2 \int \rho(\boldsymbol{r}) \sum_i \phi_i(\boldsymbol{r}) \left[\log\left(\bar{\rho}(\boldsymbol{r})\phi_i(\boldsymbol{r})\right) - 1\right] d\boldsymbol{r}.$$

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Therefore,

$$-S \doteq c^2 \sum_j m_j \sum_i \phi_i(\boldsymbol{r}_j) \left[\log\left(\frac{\rho_j \phi_i(\boldsymbol{r}_j)}{m_j}\right) - 1\right]$$

Building a theory: energy part, constraint

The energy term is

$$E = \beta \int \rho(\boldsymbol{r}) \sum_i \phi_i(\boldsymbol{r}) |\boldsymbol{r} - \boldsymbol{r}_i|^2 d\boldsymbol{r}$$

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For a Gaussian kernel, $\beta r_{ij}^2 \propto -\log W_{ij}$, so for any kernel

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Also, a C0 constraint,

$$L_{\mathrm{C0}} = \sum_j m_j \alpha_j \left(\sum_i \phi_i(\boldsymbol{r}_j) - 1 \right)$$

Density, revealed

Before including the pressure: extremezing $F = E - S$ with respect to $\phi_i(\boldsymbol{r}_j)$ and α_j,

$$F'' = -c^2 \sum_j m_j \left[\log\left(\frac{m_j Z_j}{\rho_j} \right) + 1 \right]$$

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But, pressure is not included, and we do not want anything happening yet! This fixes the density:

$$\rho_j := m_j \sum_i W_{ij}$$

Building a theory: pressure

Finally, there will be a pressure term

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$$V_i = \int \phi_i(\boldsymbol{r}) d\boldsymbol{r} \doteq \sum_j \frac{m_j}{\rho_j} \phi_i(\boldsymbol{r}_j).$$

The C0 model: shape functions

$$\phi_i(r_j) = \frac{W_{ij} z_{i,j}}{Z_j}, \qquad z_{i,j} := \exp\left(\frac{p_i}{\rho_j c^2}\right)$$

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$$f_i^{(3)} = -\sum_j \left(\frac{m_j^2 \bar{p}_j}{\rho_j^2} + \frac{m_i^2 \bar{p}_i}{\rho_i^2} \right) \nabla_i W_{ij}, \qquad \bar{p}_i := \sum_j \phi_j(r_i) p_j.$$

The C0 model: pressure closure

Since the volumes explicitely depend on the pressures, $V_i = \sum_j \frac{m_j}{\rho_j} \phi_i(\boldsymbol{r}_j)$, we may look for the pressure field that fixes their values, through a Newton-Raphson method:

$$\sum_j \left(\frac{\partial V_i}{\partial p_j} \right)^{(n)} \left[p_j^{(n+1)} - p_j^{(n)} \right] = - \left(V_i - V_i^0 \right)^{(n)} ,$$

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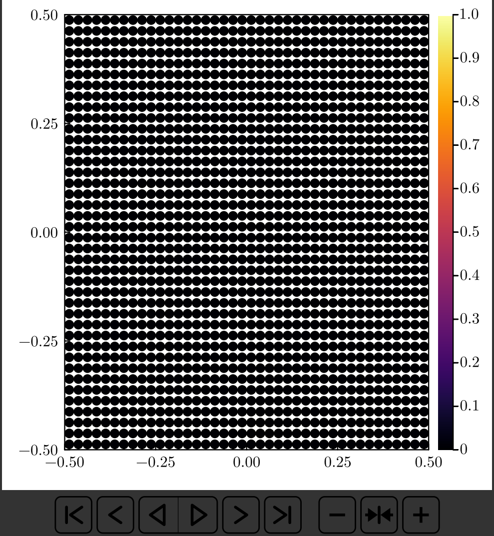
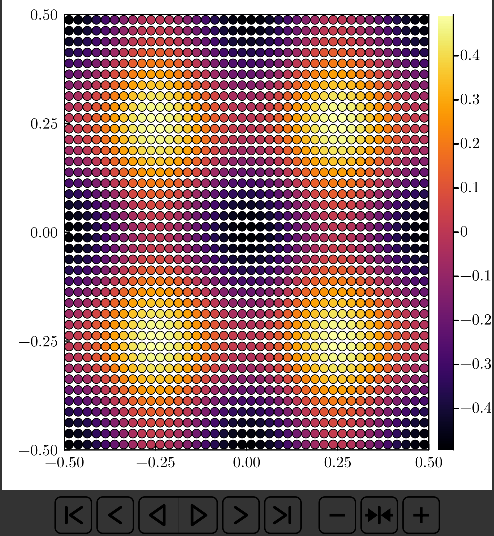
this is a Poisson equation to solve for the pressures.

TG vortex sheet



TG vortex sheet, SPH



Future: C1 shape functions

$$L_{\mathrm{C1}} = \sum_j m_j \boldsymbol{\lambda}_j \cdot \left(\sum_i \phi_i(\boldsymbol{r}_j)\boldsymbol{r}_i - \boldsymbol{r}_j \right)$$

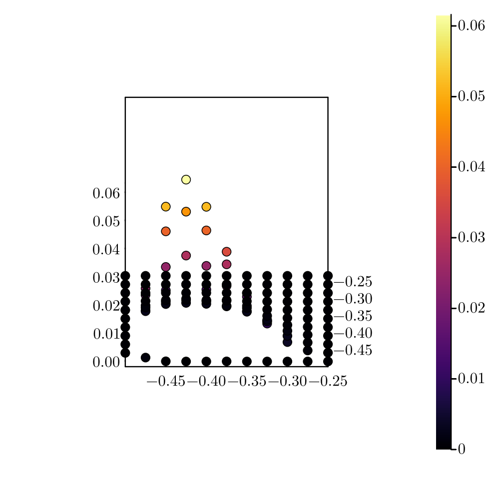
Future: C1 shape functions

$$L_{\mathrm{C1}} = \sum_j m_j \boldsymbol{\lambda}_j \cdot \left(\sum_i \phi_i(\boldsymbol{r}_j)\boldsymbol{r}_i - \boldsymbol{r}_j \right)$$

SFs are now C1 consistent, there is a new force term, but most importantly, perhaps ...

C1 shape functions

The SFs have a sort of boundary detection, and have a "weak-Kronecker condition".



Thanks, and Ref.

Thanks for your attention.

[1] Daniel Duque, *A unified derivation of Voronoi, power, and finite-element Lagrangian computational fluid dynamics*, European Journal of Mechanics - B/Fluids **98**, 268-278 (2023)

Extra: SFs as interpolants

Consider the smoothing

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Then C0, $\sum_i \phi_i(\boldsymbol{r}_j) = 1$, means constant fields have proper constant smoothing.
While C1, $\sum_i \phi_i(\boldsymbol{r}_j)\boldsymbol{r}_i = \boldsymbol{r}_j$ means linear fields have proper linear smoothing.