Convergence studies in application cases with SPH

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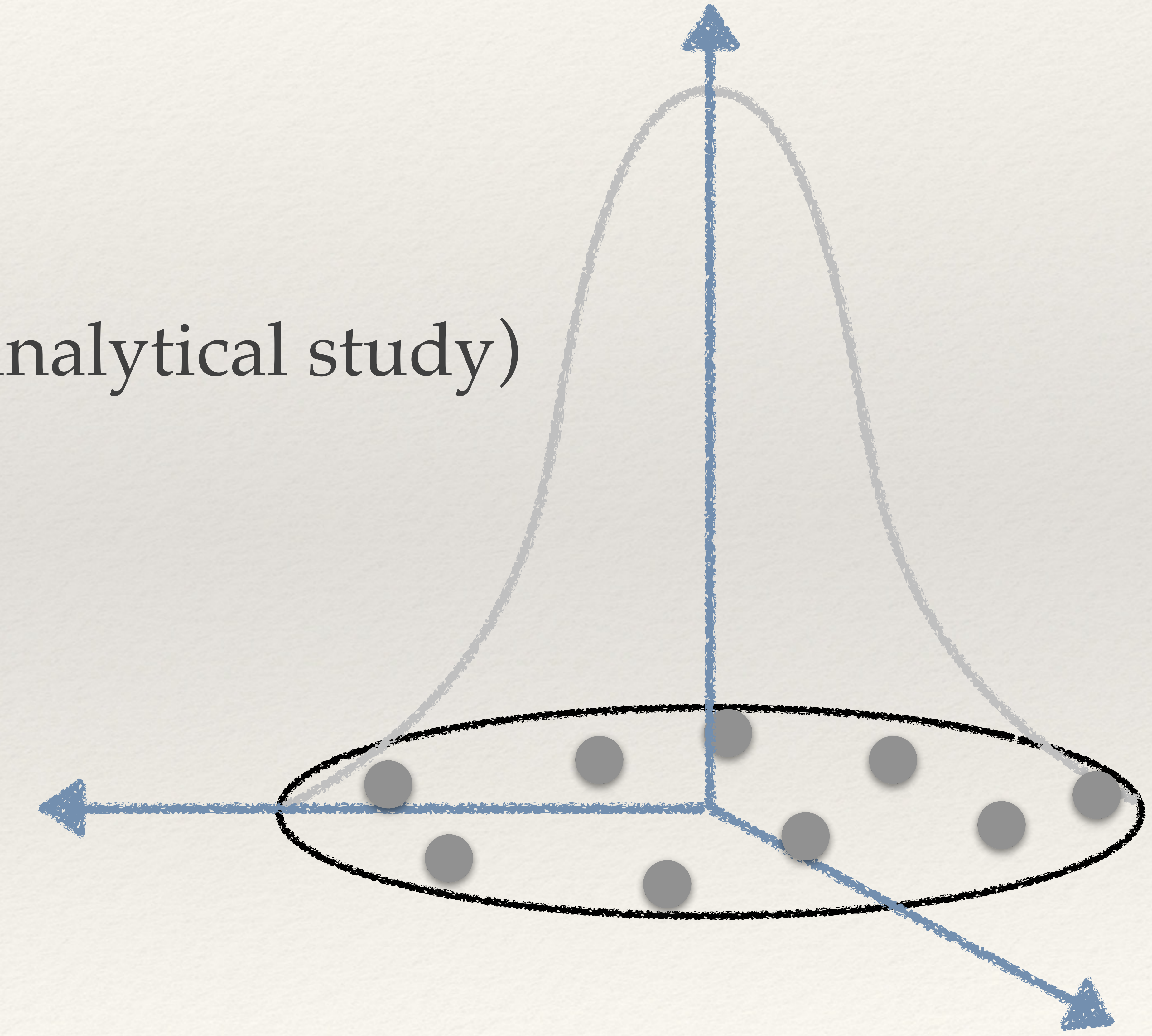


Introduction

- Convergence is one of the most relevant topics in any numerical method. In the context of SPH: SPHERIC **Grand Challenge 1: Convergence Consistency and Stability**
- Convergence is difficult to study, from an analytical perspective. Simple cases can be picked at a first stage
- Regarding convergence in SPH, there are few studies exist in the literature [Franz & Wendland 2018]. Not **diffusive terms** were considered therein
- These studies consider convergence at the discrete level. A possible approach is to study the convergence at the integral level

[Franz & Wendland 2018] Tino Franz and Holger Wendland. Convergence of the Smoothed Particle Hydrodynamics Method for a Specific Barotropic Fluid Flow: Constructive Kernel Theory. SIAM Journal on Mathematical Analysis, 50(5):4752–4784, 2018

Introduction

- What is the link between them? From [Sanz-Serna, 1985]:

$$AU = b$$

$$AU - Au = b - Au$$

$$A^{-1}\left(AU - Au\right) = A^{-1}\left(b - Au\right)$$

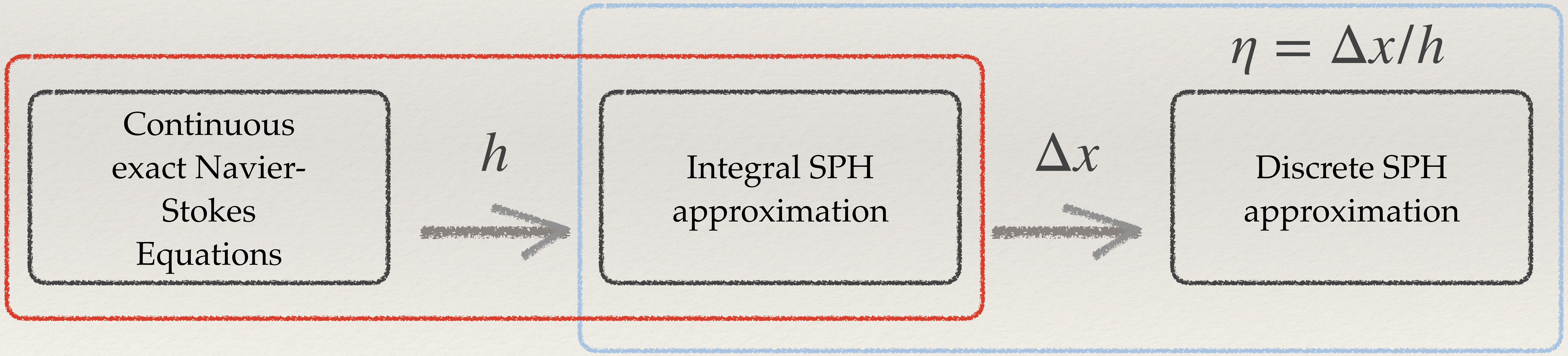
$$\|U - u\| \leq \|A^{-1}\| \, \|b - Au\|$$

[Sanz-Serna, 1985] Jesus Maria Sanz Serna. Stability and convergence in numerical analysis - I: Linear problems - a simple comprehensive account.

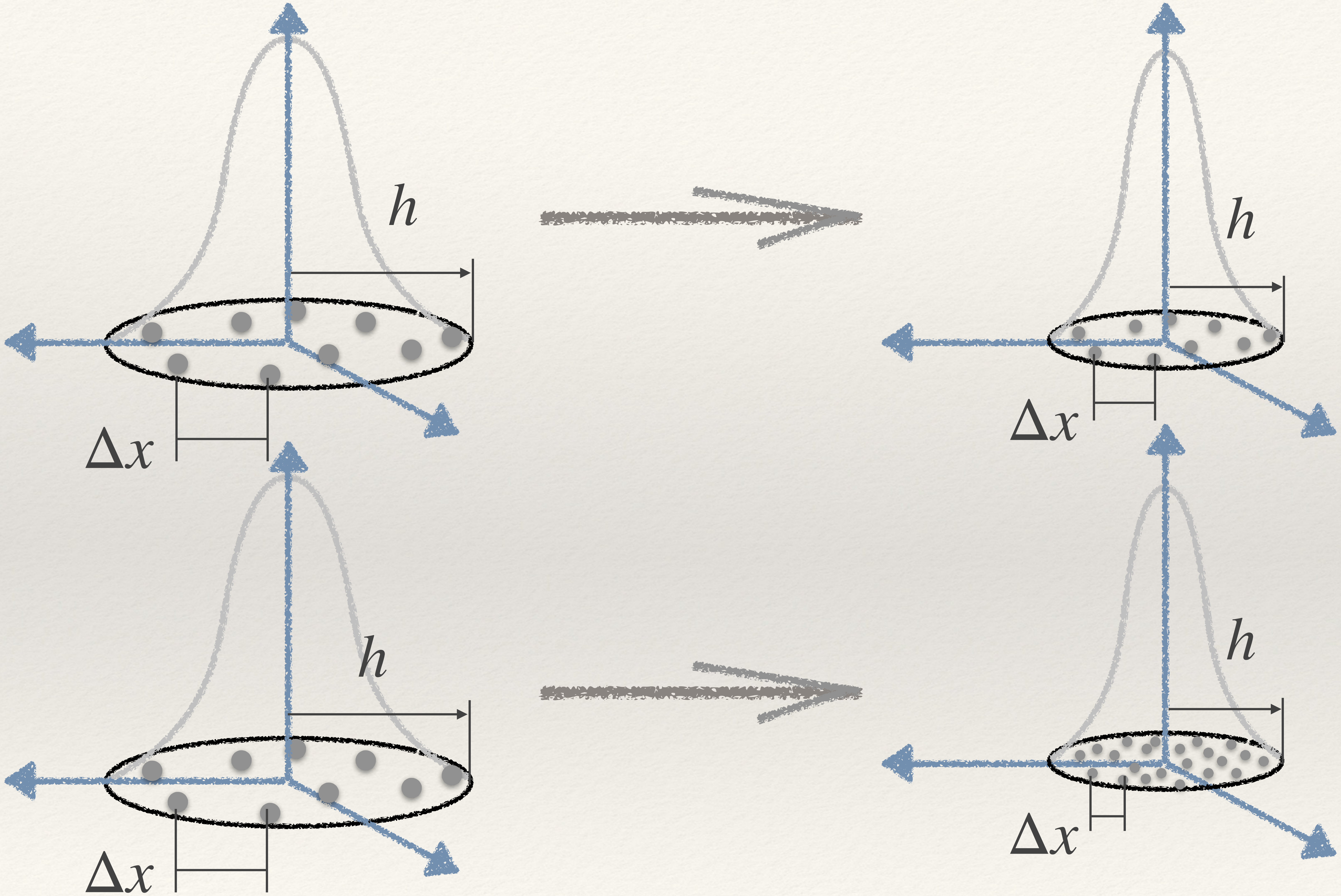
Introduction: SPH

- SPH can be regarded as a two-step approximation method (smoothing length, inter-particle distance)

$$f \simeq \ll f \gg_h = \int_\Omega f(\mathbf{y})\, W_h(\mathbf{x}-\mathbf{y})\, \mathrm{d}\mathbf{y} \qquad \ll f \gg_h \simeq \langle f \rangle(\mathbf{x}) = \sum_{j=1}^{N} f(\mathbf{x})\, W_h(\mathbf{x}-\mathbf{x}_j)\, V_j$$



Introduction: SPH



$$h \longrightarrow 0$$

$$\Delta x \longrightarrow 0$$

$$\eta = \Delta x / h = cst$$

$$h \longrightarrow 0$$

$$\Delta x \longrightarrow 0$$

$$\eta = \Delta x / h \longrightarrow 0$$

Methodology

- Fourier transform is used to study the convergence. The first goal is to obtain the Fourier transform of the main operators

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(\mathbf{x})\, e^{-i\xi\mathbf{x}}\, \mathrm{d}\mathbf{x}$$

- The SPH operators can be expressed as convolutions:

$$\int_{\mathbb{R}^d} u(\mathbf{y}) W_h(\mathbf{x}-\mathbf{y}) d\mathbf{y} = \left(u * W_h\right)(\mathbf{x})$$

- Convergence is established through showing consistency and stability
- The SPH Kernel is defined as usual, with W being integrable, differentiable, of compact support and normalized

$$W_h(\mathbf{x}) := \frac{1}{h^d} W\left(\frac{\mathbf{x}}{h}\right) \qquad W(\mathbf{x}) := \tilde{W}(|\mathbf{x}|) \qquad \int_{\mathbb{R}^d} W_h(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1$$

- A positive Fourier transform is also required [Wendland 1995, Dehnen & Aly 2012]

[Wendland 1995] Holger Wendland, Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. Adv. Comput. Math., 4(4): 389–396

[Dehnen & Aly 2012] Walter Dehnen and Hossam Aly. Improving convergence in smoothed particle hydrodynamics simulations without pairing instability. Monthly Notices of the Royal Astronomical Society

Hydrostatic problem: convergence

- The focus is set on the following integral equation, representing the integral SPH discretization of the ODE constituting the 1D hydrostatic problem

$$p'(x) = -1, \quad x \in (-\infty, 0), \quad p(0) = 0 \qquad \ll p' \gg_T = \int_{-\infty}^{0} p_h(y)\, W_h'(x-y)\, \mathrm{d}y = -1, \quad x \in (-\infty, 0)$$

- A bound for the error is obtained [Macià 2020]:

$$\left(\int_{-\infty}^{\infty} |\, \widehat{p}(\xi) - \widehat{p_h^-}(\xi)\, |^2 \, \mathrm{d}\xi \right)^{1/2} \leq \left(\int_{-\infty}^{\infty} \frac{(1 - \widehat{W}(h\xi))^2}{\xi^4} \mathrm{d}\xi \right)^{1/2}$$

$$\int_{-\infty}^{\infty} \widehat{W}(h\xi)\, |\, \widehat{p}(\xi) - \widehat{p_h^-}(\xi)\, |^2 \, \mathrm{d}\xi \simeq \int_{-\infty}^{\infty} |\, \widehat{p}(\xi) - \widehat{p_h^-}(\xi)\, |^2 \, \mathrm{d}\xi$$

- It is not clear wether this condition is satisfied by the actual solution in the limit $\eta \longrightarrow 0$, while numerical solutions converge in general when η is kept constant [Merino-Alonso 2020]

[Macià 2020] F. Macià, P.E. Merino-Alonso and A. Souto-Iglesias. On the truncated integral SPH solution to the Hydrostatic problem. Computational Particle Mechanics 2020.

[Merino-Alonso 2020] Pablo Eleazar Merino Alonso, Fabricio Macià and Antonio Souto Iglesias. On the numerical solution to the truncated discrete SPH formulation of the hydrostatic problem. Journal of hydrodynamics 2020.

The advection-diffusion equation

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- SPH diffusion equations have been previously studied [Violeau 2018] [Basa et al. 2009]
- Transport velocity $\mathbf{v}$ is either constant in space and time either it represents a rigid rotation

$$\begin{cases} \partial_t u(t,\mathbf{x}) = -\mathbf{v}\cdot\nabla u(t,\mathbf{x}) + \alpha\,\Delta u(t,\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d,\ t\in(0,\infty) \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) \end{cases}$$

- The SPH version of the equation:

$$\begin{cases} \partial_t u_h(t,\mathbf{x}) = -\mathbf{v}\cdot \ll \nabla u_h \gg (t,\mathbf{x}) + \alpha \ll \Delta u_h \gg (t,\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d,\ t\in(0,\infty) \\ u_h(0,\mathbf{x}) = u_0(\mathbf{x}) \end{cases}$$

$$u_h(t,\mathbf{x}) = \left(\widehat{u_0}(\xi)\, e^{-t(i(\mathbf{v}\cdot\xi)w(|h\xi|)+\alpha|\xi|^2\Phi(|h\xi|))}\right)^{\vee}(\mathbf{x})$$

[Violeau 2018] Damien Violeau, Agnes Leroy, Antoine Joly, and Alexis Hérault. Spectral properties of the SPH laplacian operator. Computers & Mathematics with Applications, 75(10): 3649–3662, 2018.

[Basa et al. 2009] Mihai Basa, Nathan J. Quinlan, and Martin Lastiwka. Robustness and accuracy of SPH formulations for viscous flow. International Journal for Numerical Methods in Fluids, 60 (10): 1127–1148, 2009

Theorem

Theorem 1: Assume that the SPH kernel satisfies (P). Let u and u_h denote respectively the solutions of Equations (I) and (II) issued from the same initial datum u_0. Then the following hold:

- If $\mathbf{u}_0$ and all its derivatives up to order two are square-integrable in $\mathbb{R}^d$ then:

$$\sup_{t\in[0,T]} ||u_h(t,\cdot) - u(t,\cdot)||_{L^2(\mathbb{R}^2)} \longrightarrow 0, \quad h \longrightarrow 0.$$

- If $\widehat{\mathbf{u}_0}$ and $|\xi|^2\,\widehat{u_0}(\xi)$ are integrable in $\mathbb{R}^d$ then:

$$\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^d} |u_h(t,\mathbf{x}) - u(t,\mathbf{x})| \longrightarrow 0, \quad h \longrightarrow 0.$$

$$w(|\xi|) := \widehat{W}(\xi)$$

$$\varphi(\rho) = \int_0^1 \tau w(\rho\tau)\,\mathrm{d}\tau > 0 \qquad \text{(P)}$$

[Macia 2022] Fabricio Macià, Pablo Eleazar Merino Alonso and Antonio Souto Iglesias. On the convergence of the solutions to the integral SPH heat and advection-diffusion equations: theoretical study and numerical verification. Computer Methods in Applied Mechanics and Engineering, 2022

Fourier representation of SPH Laplacian

$$\ll \Delta u_h \gg_M (\mathbf{x}) := \frac{2}{h^2} \int_{\mathbb{R}^d} (u_h(\mathbf{y}) - u_h(\mathbf{x}))\, G_h(|\mathbf{x} - \mathbf{y}|)\, \mathrm{d}\mathbf{y}$$

$$\widehat{\ll \Delta u_h \gg_M} (\mathbf{x}) = -2|\xi|^2 \varphi(|h\xi|) \hat{u}(\xi)$$

$$e_h(t,\xi) := \widehat{u_h}(t,\xi) - \hat{u}(t,\xi)$$

Consistency

$$|e_h(t,\xi)| \leq h^2 C(1 + \alpha|\xi|^2)|\widehat{u_h}(t,\xi)|$$

Stability

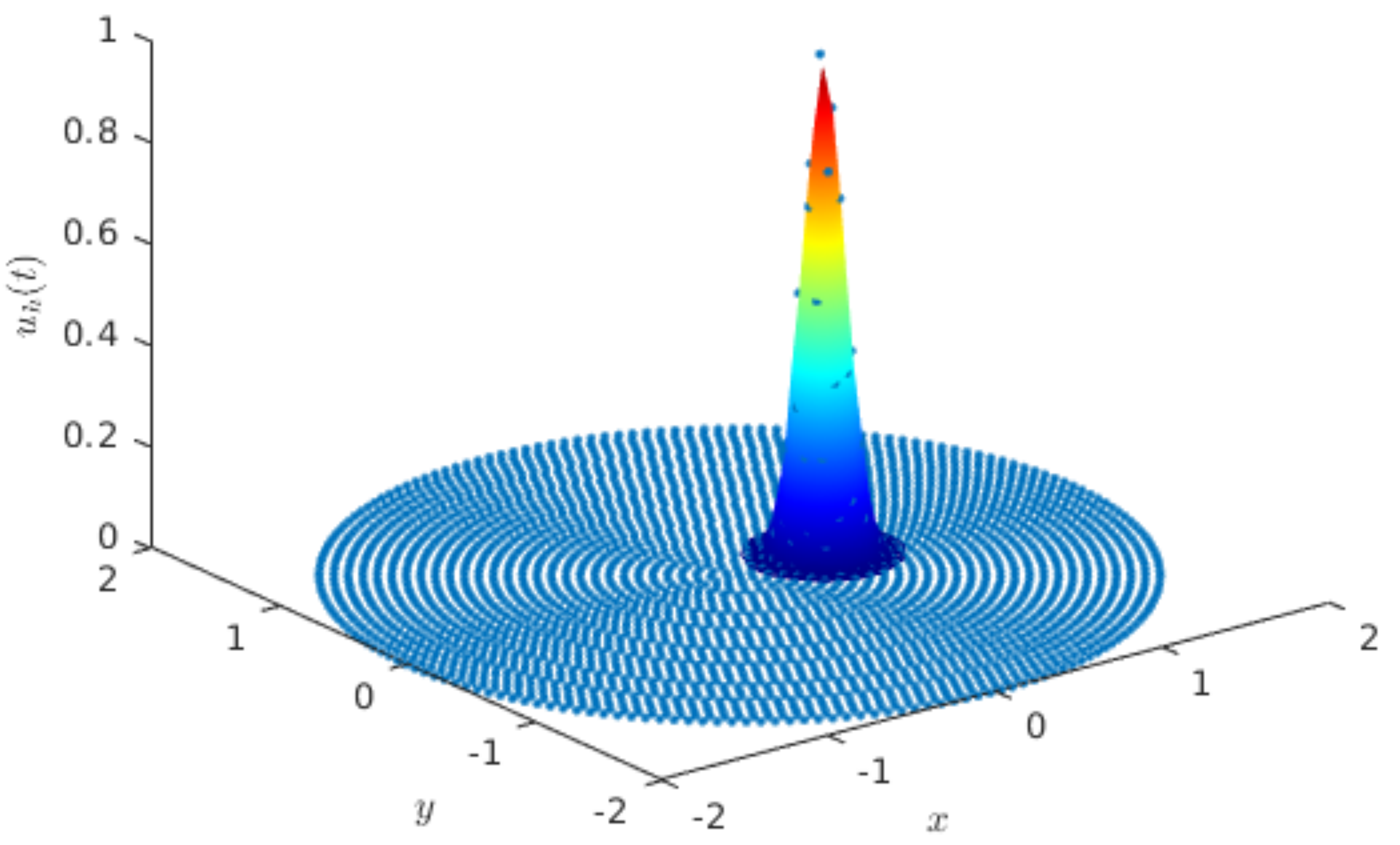
$$|\widehat{u_h}(t,\xi)| = \widehat{u_0}(\xi)\, e^{-t\alpha|\xi|^2\varphi(|h\xi|))} |\widehat{u_0}(\xi)|$$

$$\varphi(\rho) = \int_0^1 \tau w(\rho\tau)\, \mathrm{d}\tau > 0$$

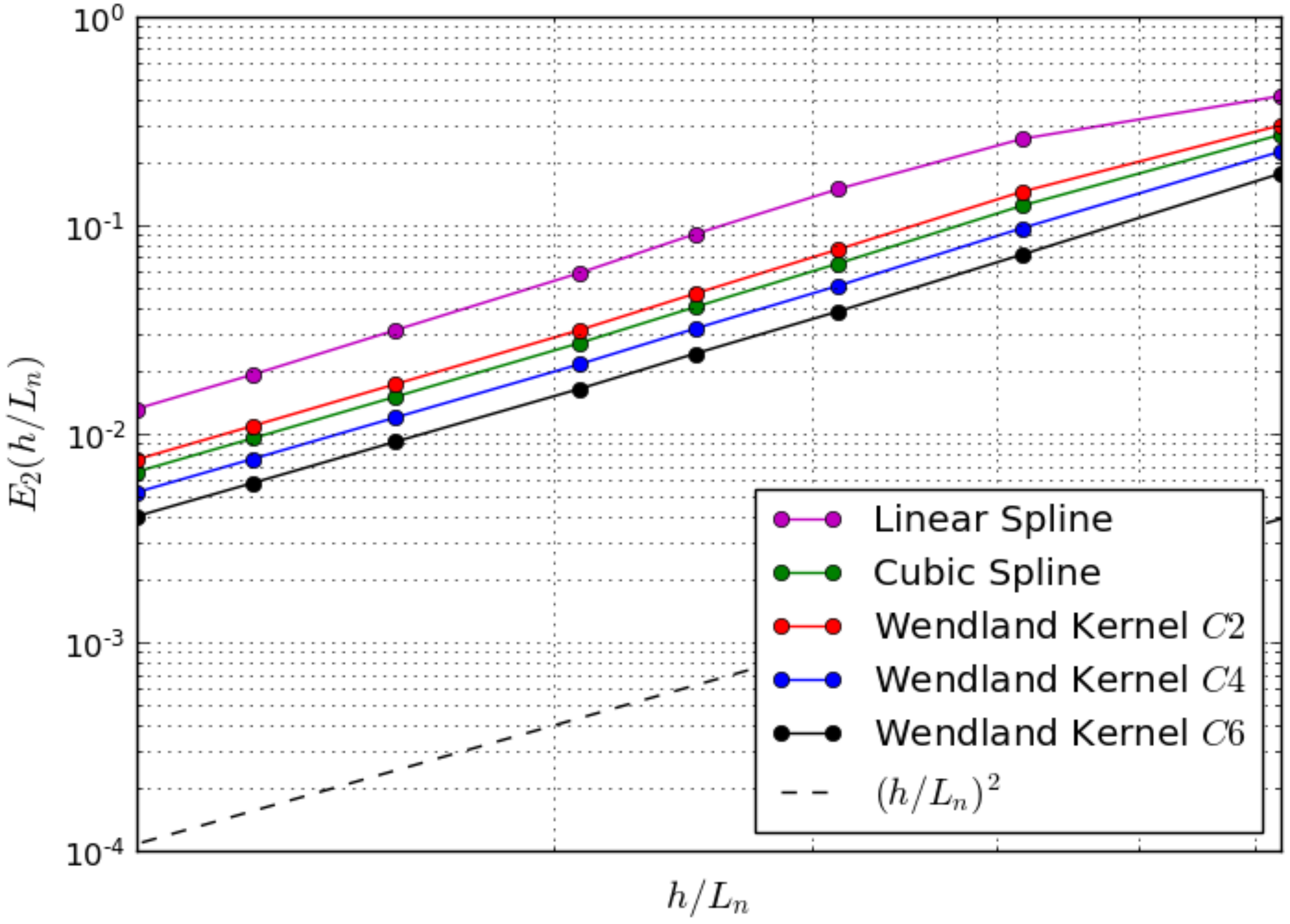
$$|\widehat{u_h}(t,\xi)| \leq |\widehat{u_0}(\xi)|$$

Numerical example: rotating Gaussian

Initial condition



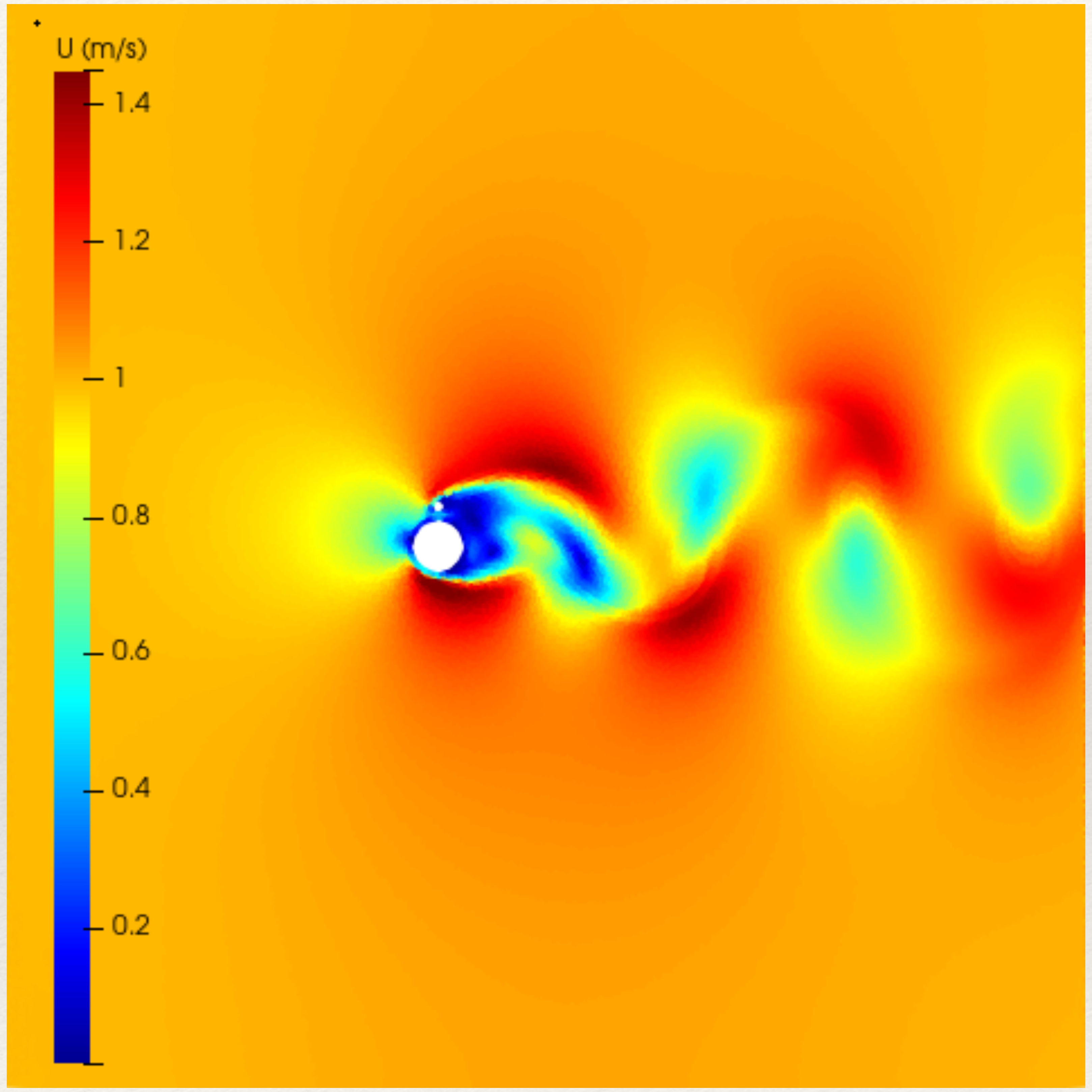
Error in terms of the smoothing length

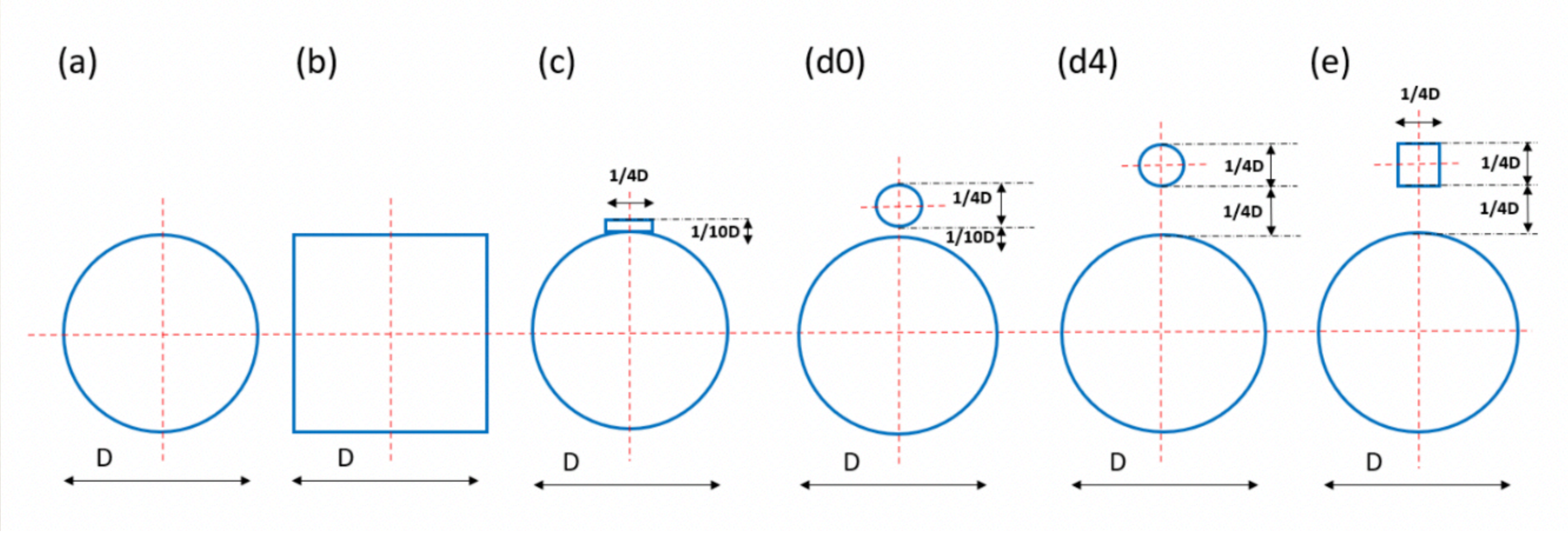


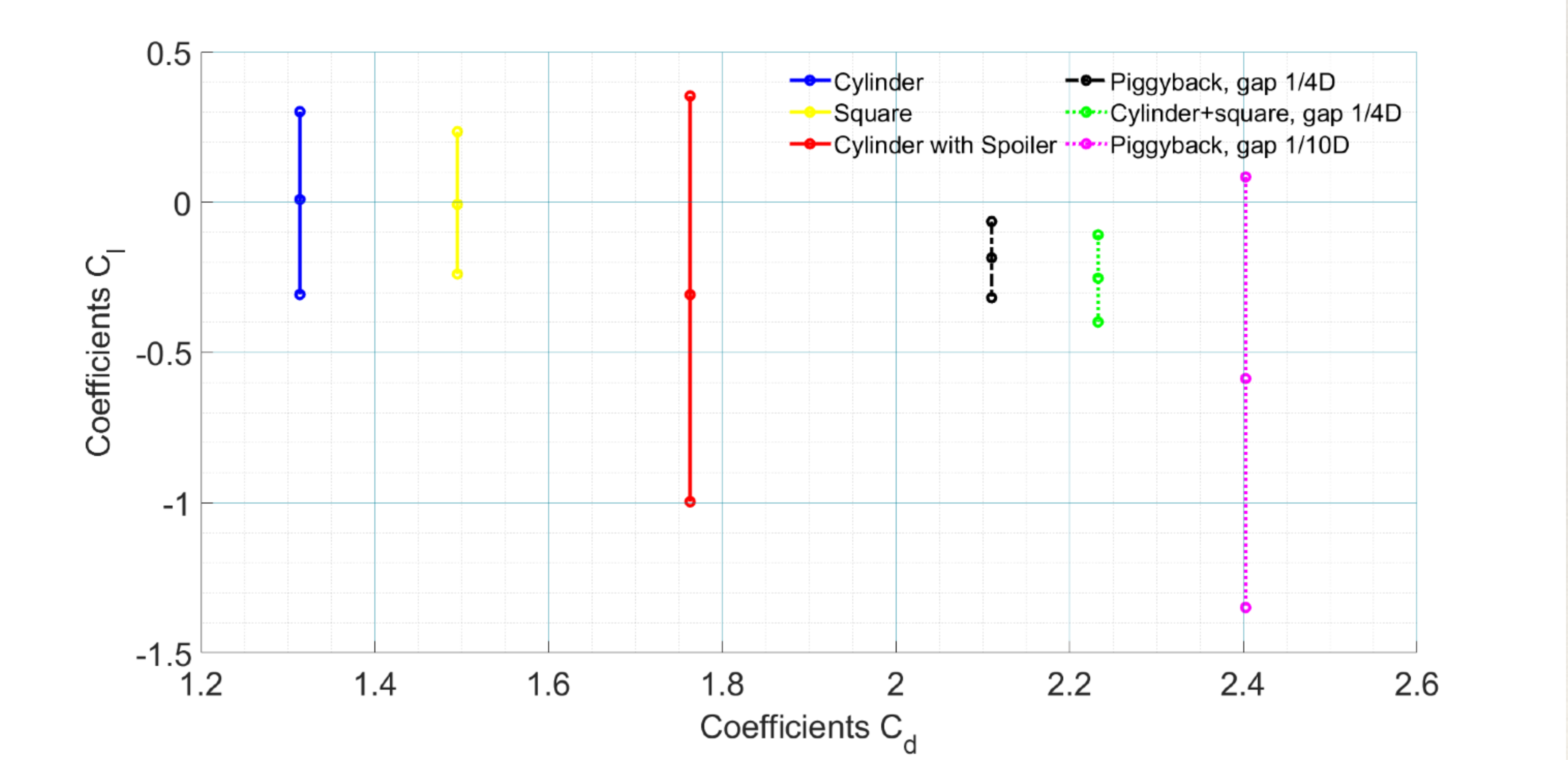
[Merino-Alonso 2022] Pablo Eleazar Merino Alonso, Fabricio Macià and Antonio Souto Iglesias. On the convergence of the solution to the integral SPH advection-diffusion equation with rotating transport velocity field. Acta Mechanica Sinica, 2022.

Lift and drag in submerged obstacle

- An application case using **AQUAgpusph**, the flow around an object with different shapes has been considered Results validated against existent results in the literature

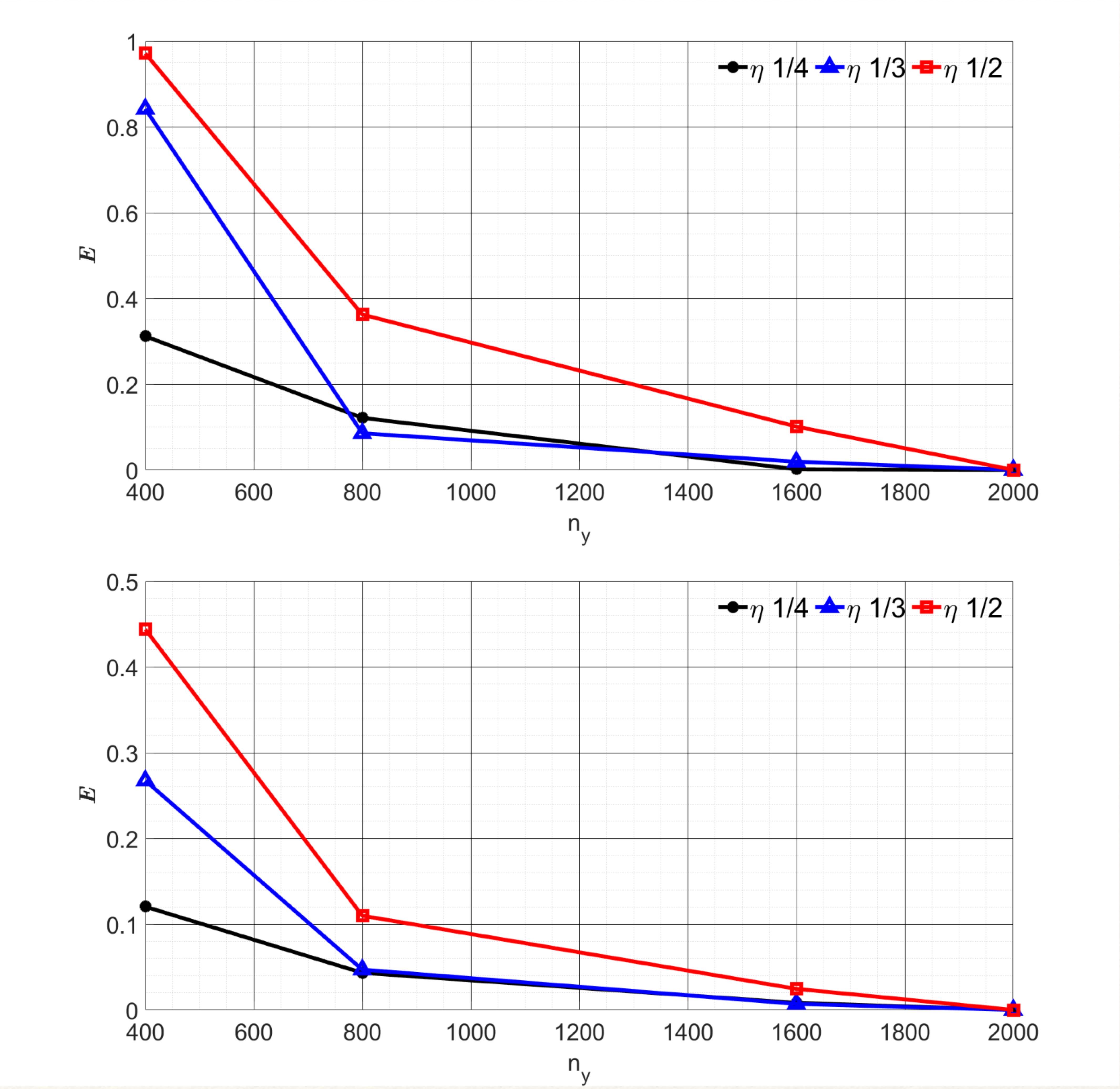




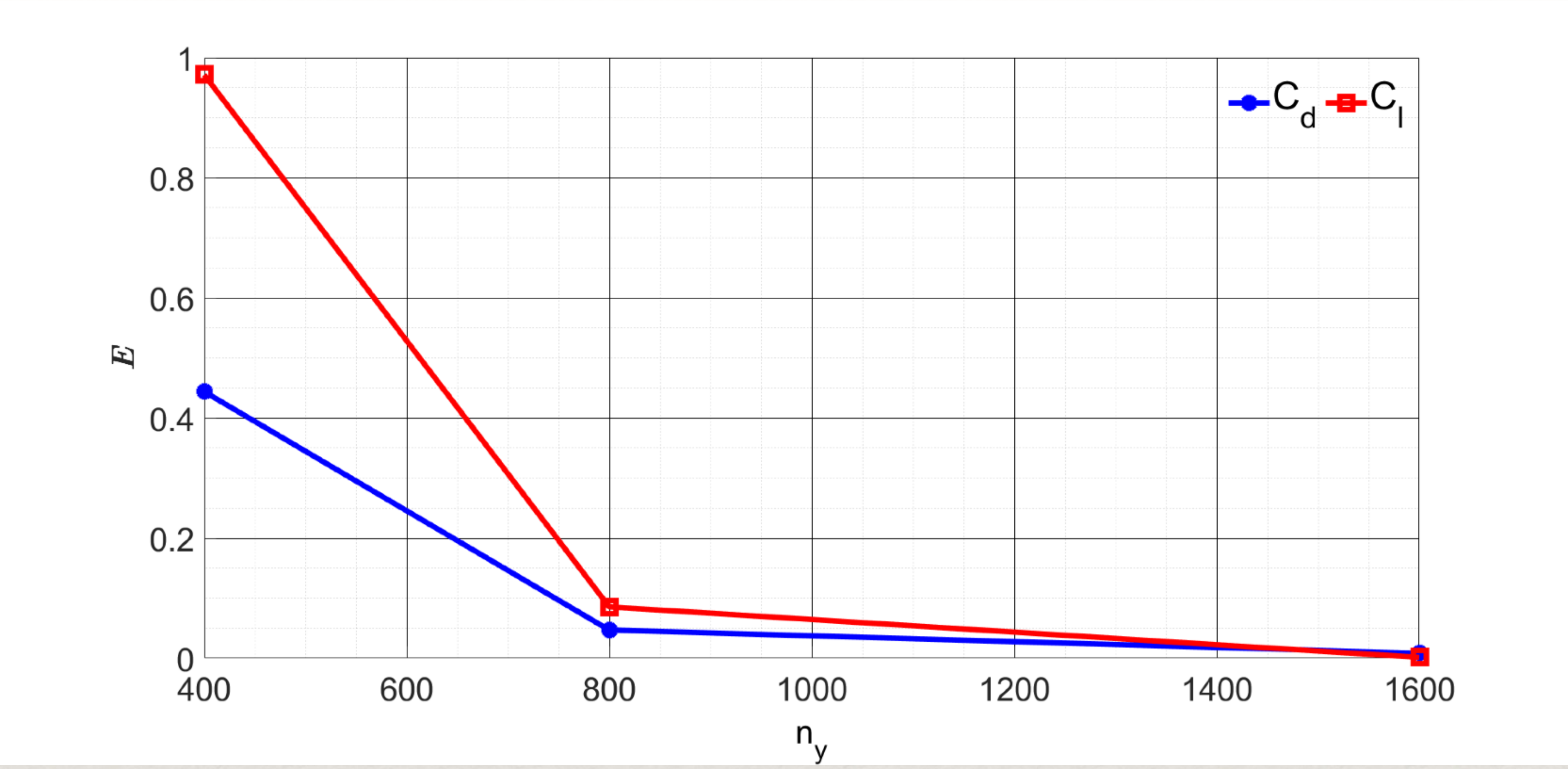


[Acosta 2024] Gustavo Fabián Acosta, Javier Calderon Sanchez, Pablo Eleazar Merino Alonso. A hydrodynamic study of various obstacle shapes in 2D using SPH. [Sent for publication]

Lift and drag in submerged obstacle



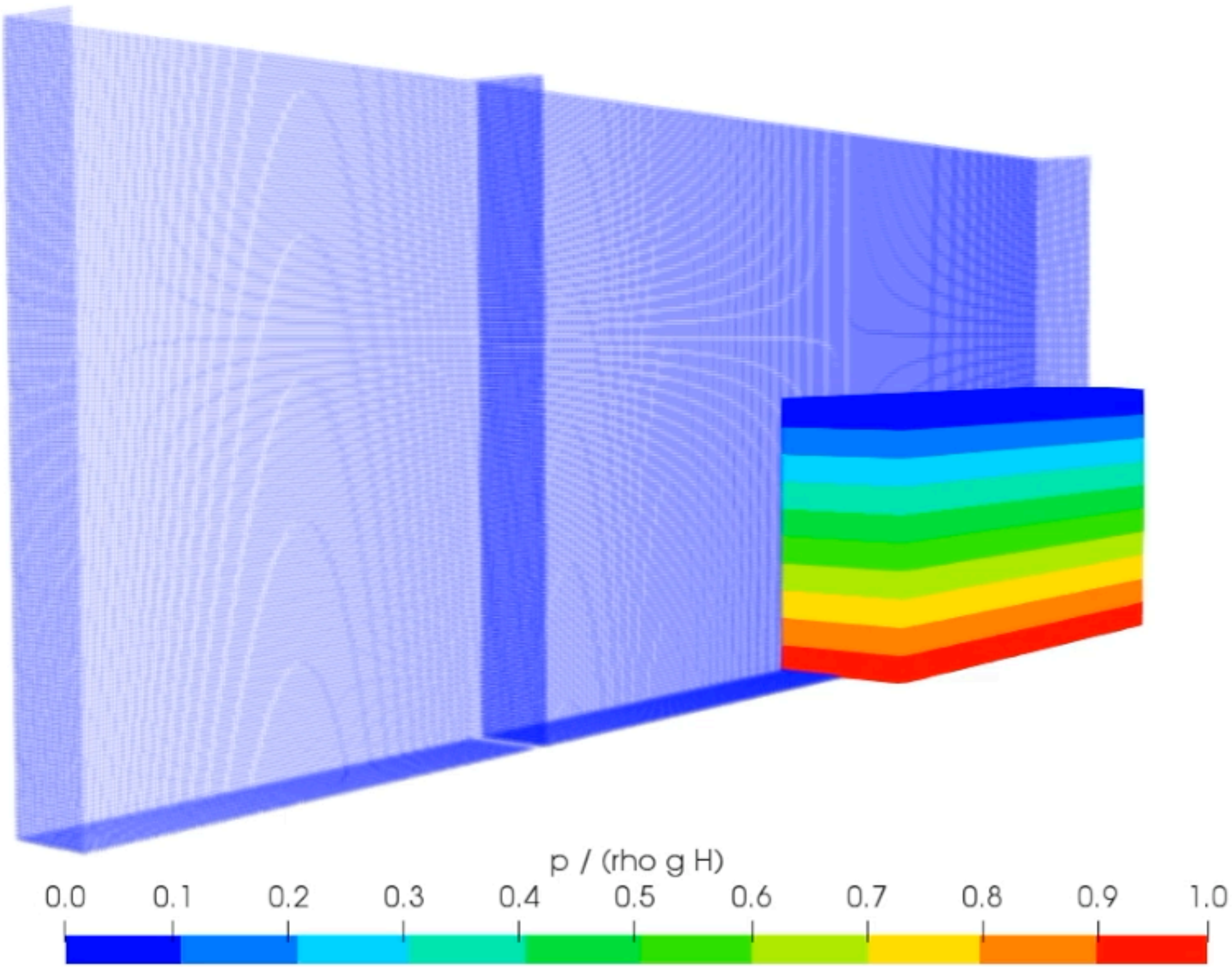
- Left: errors in lift (top) and drag (bottom) coefficients with η constant. Bottom, same with $\eta \longrightarrow 0$



Case	dr/D	h/D
$(n_y = 400,\ \eta = 1/2)$	0,05	0,1
$(n_y = 800,\ \eta = 1/3)$	0,025	0,075
$(n_y = 1600,\ \eta = 1/4)$	0,0125	0,05

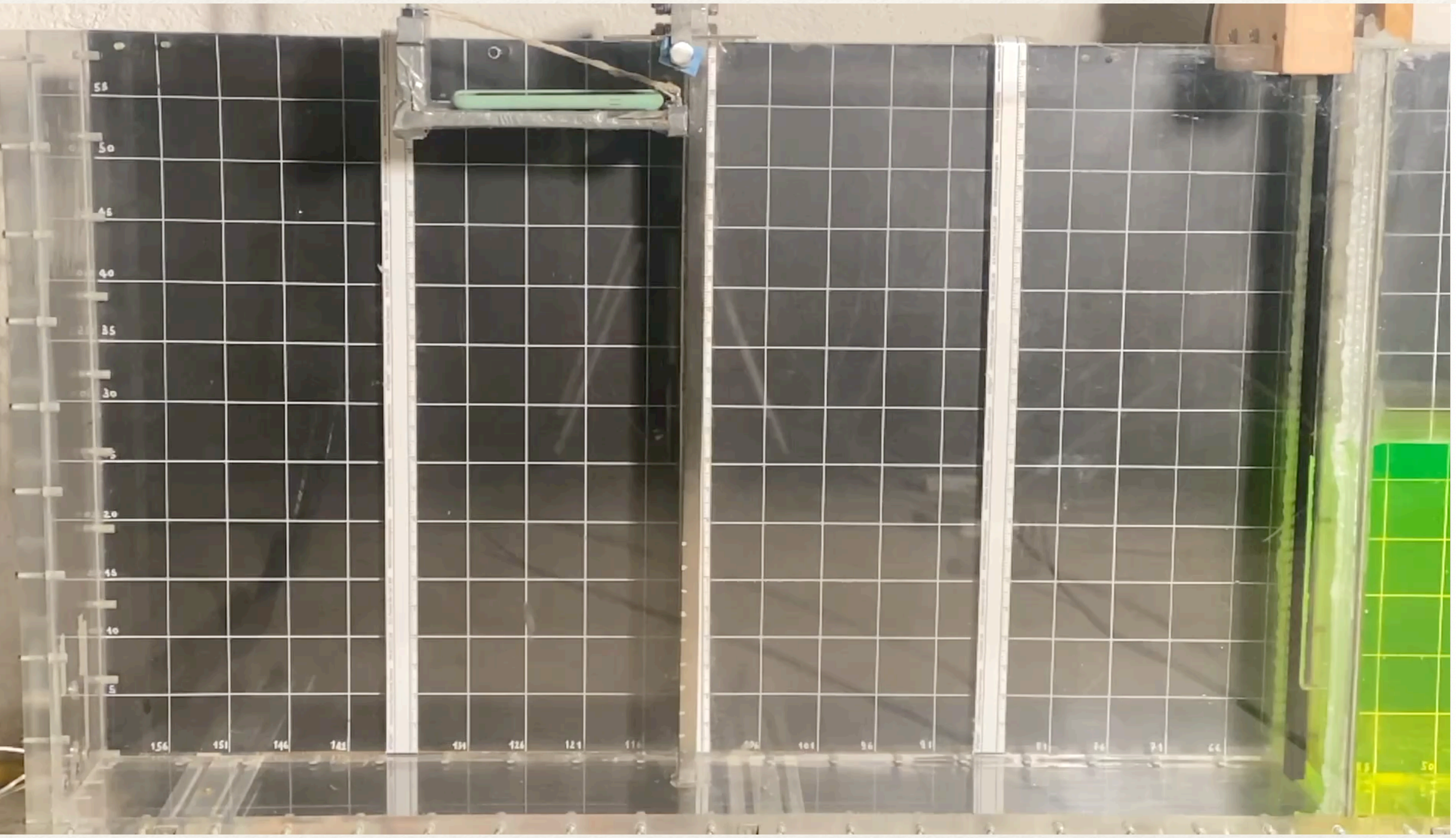
AQUAgpusph

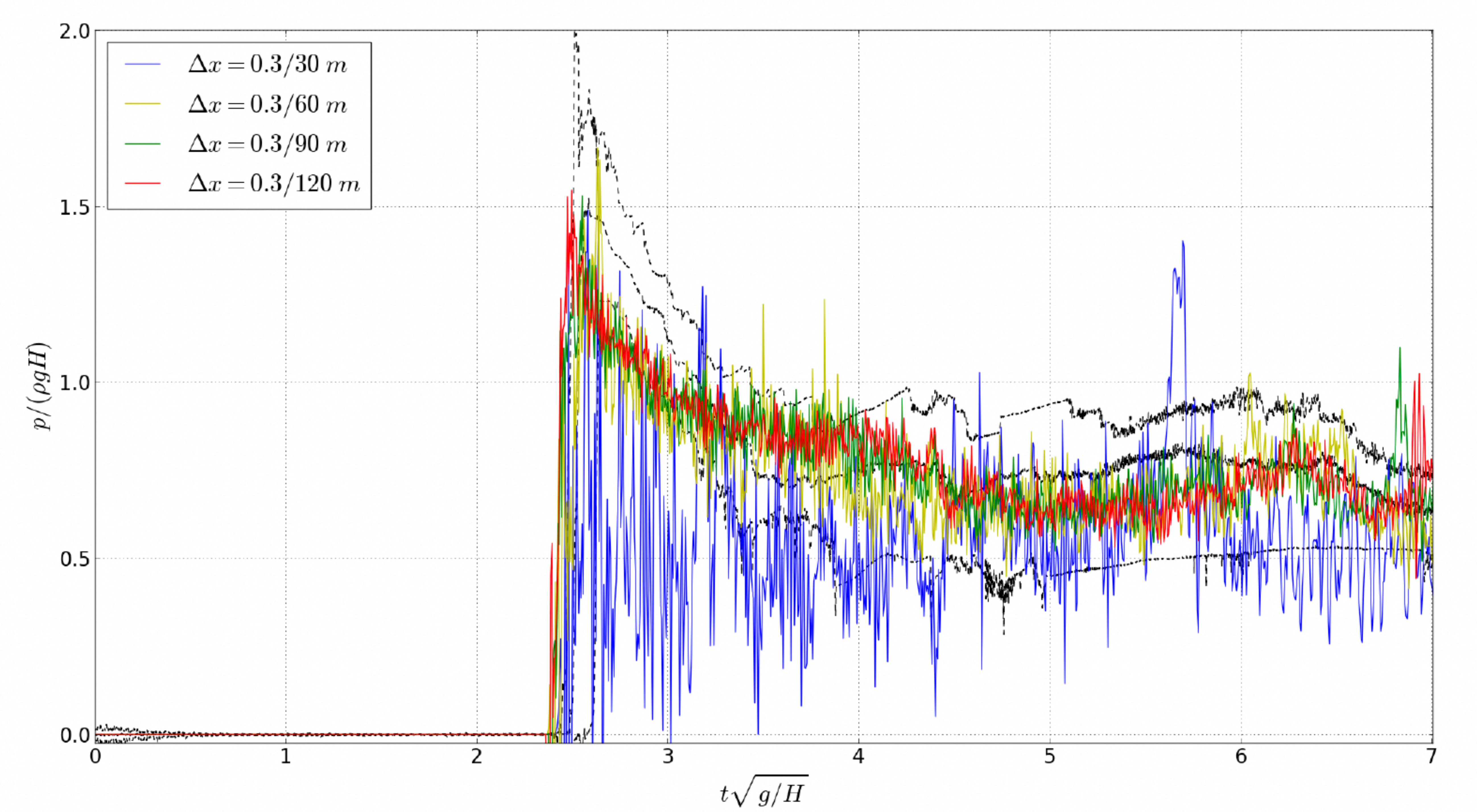
Time: 0.00



Dam Break

- Experimental results have been used to validate the simulations
- Numerical solutions with **AQUAgpusph** have been compared against experiments





[Martinez-Carrascal 2024] Jon Martinez Carrascal, Pablo Eleazar Merino Alonso, Ignacio Mengual Berjon, Mario Amaro, Antonio Souto Iglesias. Experimental study of the pressure loads in obstacle-induced 3D flows during a dam break. [To be sent for publication]

Conclusions & Future Work

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- The convergence of the integral SPH solutions to **SPH hydrostatic** and **advection-diffusion** equations have been analytically established
- In the case of the advection-diffusion equation, convergence holds when the kernel satisfies the **positivity condition**. This condition is weaker than the requirement of having a **positive Fourier Transform**
- The problem of the flow around an obstacle has been studied using **AQUAgpusph**. The convergence has been studied numerically. It is shown that the lift and drag forces converge both when the inter-particle distance over smoothing length ratio is kept constant and when it tends to zero
- The Dam Break case has been reproduced using **AQUAgpusph** and compared qualitatively against experimental results
- Future Work: Perform a systematic study of the convergence in the dam break case
- Future Work: Extend the analytical studies to more complex cases including that where the inter-particle distance over smoothing length ratio is kept constant

Thanks a lot for your attention
Questions?

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